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PREFERENTIAL VOTING.

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Let there be given three candidates A, B, and C. Let there be S voters and suppose each voter expresses his first, second, and third choice. Suppose the voting to have been done. It is the object of this paper to determine the conditions under which it is possible to assign weights to the votes for first, second, and third choice so that any preassigned candidate may win.

Let the votes which each candidate receives be exhibited by the following table:

	Choice		
	1st	2d	3d
A	A_1	A_2	A_3
B	B_1	B_2	B_3
C	C_1	C_2	C_3

In this table it is to be noted that

$$A_i + B_i + C_i = S \qquad (i = 1, 2, 3),$$

$$A_1 + A_2 + A_3 = B_1 + B_2 + B_3 = C_1 + C_2 + C_3 = S.$$

Let x, y, and z be the weights assigned to first, second, and third choice with the condition

$$x > y > z.$$

Then the number of points received by A in this contest is given by $A_1x+A_2y+A_3z$, with similar expressions for the number of points received by B and C.

Consider now the three planes

(A) $$A_1x + A_2y + A_3z = 0,$$

(B) $$B_1x + B_2y + B_3z = 0,$$

(C) $$C_1x + C_2y + C_3z = 0.$$

These three planes form a trihedral, unless two or more coincide, with vertex at the origin. Every point in the first octant is within the same one of the eight compartments of space marked off by these three intersecting planes. The three planes

(AB) $$(A_1 - B_1)x + (A_2 - B_2)y + (A_3 - B_3)z = 0,$$

(AC) $$(A_1 - C_1)x + (A_2 - C_2)y + (A_3 - C_3)z = 0,$$

(BC) $$(B_1 - C_1)x + (B_2 - C_2)y + (B_3 - C_3)z = 0,$$

are the loci of points for which respectively A and B, A and C, or B and C are tied. These three planes are co-axial, the axis of the pencil being the line

$$x = y = z.$$

Cut now the system of six planes which we have by the plane

(1) $$x + y + z = \text{any real positive constant.}$$

Denote the traces of the planes A, B, C, AB, AC, BC on the plane (1) by the symbols a_1, b_1, c_1, a_1b_1, a_1c_1, b_1c_1 respectively (see Fig. 1).

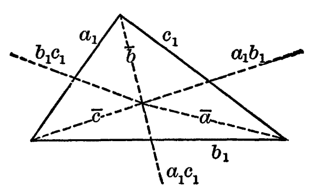

Fig. 1.

The line a_1b_1 and the origin determine a plane which we shall call the plane a_1b_1. In like manner the planes a_1c_1, b_1c_1, and a_1, b_1, c_1 are determined.

Let us denote the region inclosed by the four planes a_1b_1, a_1c_1, b_1, c_1 by $\bar{a}$; likewise, b_1c_1, b_1a_1, a_1, c_1 by $\bar{b}$; and c_1a_1, c_1b_1, a_1, b_1 by $\bar{c}$. It is clear that in each of the regions $\bar{a}$, $\bar{b}$, $\bar{c}$ there are points for which x, y, z are all positive. If now we choose as weights the coördinates of any point in the region $\bar{a}$ ($\bar{b}$ or $\bar{c}$) then A (B or C) is the successful candidate. It remains to show under what conditions it is impossible to find points in $\bar{a}$ ($\bar{b}$ or $\bar{c}$) for which

$$x > y > z.$$

If two candidates A and C receive an equal number of votes for first and second choice they are tie under any system of weights. It becomes then a question as to whether B can win over A. Let *A and B receive the same number of votes for* *first choice while A receives the most votes for second choice.* Then

$$B_3 = A_2 \qquad \text{and} \qquad B_2 = A_3.$$

It is now impossible for B to win. For if B is to win we must have

$$xA_1 + yA_2 + zA_3 < xA_1 + yA_3 + zA_2,$$

which reduces to

$$y < z$$

and this is contrary to hypothesis. In like manner we can show that *if A and B have an equal number of votes for second or third choice and A has the greater number of votes for first choice, then it is impossible for B to win.*

There remains the case where any two candidates A and B receive an unequal number of votes for first, second, and third choice. We will now suppose that A receives the most votes for first choice. Under this hypothesis it is clear that the x-axis lies within the region $\bar{a}$, for the weights 1, 0, 0 are such as to make A win.

Designate by ab, ac, bc the lines of intersection of the plane

(2) $$x = \text{any real positive constant}$$

with the planes AB, AC, BC. Designate by O' and P its intersections with the x-axis and with the line $x = y = z$. (See figures 2 and 3.) Let $O'z'$ and $O'y'$

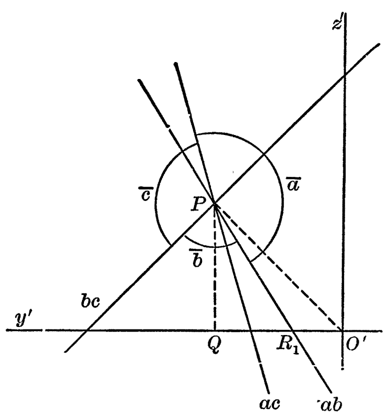

Fig. 2.

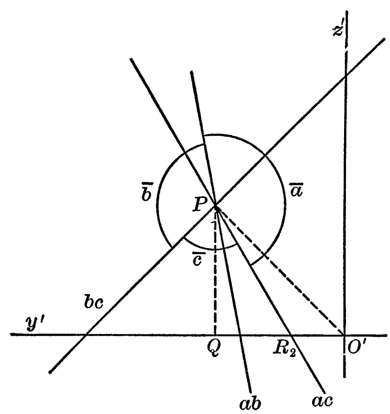

Fig. 3.

be the intersections of the plane (2) with the planes $y = 0$ and $z = 0$. Draw QP perpendicular to $O'y'$. In order that $x > y > z$, we must choose as weights the coördinates of some point within the triangle $O'PQ$. Since the plane AB contains the line $x = y = z$, the line ab must contain the point P. The region $\bar{b}$ and the x-axis are on opposite sides of the plane AB. Hence in order that B may win over A it is necessary that the line ab have a positive slope greater than unity but not infinite. In like manner in order that C may win over A

it is necessary that the line ac have a positive slope greater than unity but not infinite.

Suppose the slopes of ab and ac both positive and greater than unity. If the slope of ab [or ac] is less than that of ac [or ab] then B [or C] may win over both A and C [or B] by choosing as weights the coördinates of any point within the region PQR_1 [or PQR_2]. From Fig. 2 (or Fig. 3) in order that C [or B] may win over B [or C] and A it is necessary that the line bc have a positive slope greater than unity but not infinite and also greater than the slope of ac [or ab].

Thus, in order that either one of B or C at will may win over the other two candidates, A having the greatest number of votes for first choice, it is necessary that each of the three lines ab, ac, bc have a positive slope greater than unity but not infinite and that of these three lines the slope of bc be the greatest.

These conditions are also sufficient. *Illustration:*

	Choice			
	1st	2d	3d	
A	9	2	9	
B	4	11	5	
C	7	7	6	Winner
Weights	4	2	1	*A*
	6	5	1	*B*
	3	2	1	*C*

A DIRECT PROOF OF DE MOIVRE'S FORMULA.

By S. LEFSCHETZ, University of Kansas.

The proposition which will be proved here, and which is practically equivalent to De Moivre's noted theorem, may be stated thus:

If X, Y, Z are three complex numbers of modulus unity, and such that their arguments x, y, z have a zero sum, then $XYZ = 1$.

Let $X = a + ia'$, $Y = b + ib'$, $Z = c + ic'$, and denote their conjugates by $\bar{X}$, $\bar{Y}$, $\bar{Z}$. The moduli being unity, we have $\bar{X} = 1/X$, $\bar{Y} = 1/Y$, $\bar{Z} = 1/Z$. Let A and B be the representative points of X and $\bar{Y}$, using Argand's diagram, O the origin, E the foot of the perpendicular AE from A to OB. As angle $BOA = (x + y)$, it follows that $OE = -c$, $EA = -c'$. The coefficients of the equations of the lines OB and AE are rational in the coördinates of A, B, that is, in a, b, a', b', and therefore the same holds for the coördinates of their intersection E. When these are known, the ratio $OE/OB = -c/1$ is determined rationally. Hence c, and similarly c', can be determined rationally with respect to a, b, a', b'. Therefore.

(1) $$Z = c + ic' = f(a, b, a', b'),$$