Consequently, SE and SF are the projections of SB and SA, respectively, on the plane DSC.

Therefore $\angle ESC < \angle BSC$, and $\angle FSC < \angle ASC$; or $\angle DSC - \angle DSE < \angle BSC$, and $\angle DSC + \angle DSF < \angle ASC$.

Adding, and remembering that $\angle DSE = \angle DSF$, $2\angle DSC < \angle BSC + \angle ASC$, $\angle DSC < \frac{1}{2}(\angle BSC + \angle ASC$.

Case II. $\angle DSC$=a right angle. (See Fig. 2.)

To prove $\angle DSC = \frac{1}{2}(\angle ASC + \angle BSC)$.

Draw DX parallel to SC, and AB perpendicular to SD.

Then SD is perpendicular to DX, and, hence, to the plane determined by AB and DX.

This plane intersects the planes of faces BSC and ASC in BE and AF, respectively.

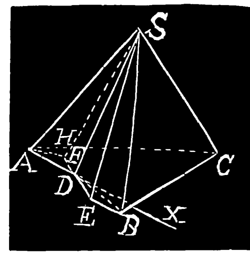


Fig. 2.

Since SC is parallel to DX, it is parallel to the plane BDX, and, hence, parallel to BE and AF.

Through SD pass a plane perpendicular to SC, intersecting the plane of AB and DX in EF, and the planes of faces BSC and ASC in SE and SF, respectively. Since CS is perpendicular to the plane FSE, so are BE and AF.

Hence, $\angle$'s BES, BED, AFS, and AFD are right angles.

Right triangles DAF and DBE are equal, since $AD=BD$ (from equality of right triangles ASD and BSD), and the vertical angles at D are equal.

Therefore $BE=AF$. Hence, since $SB=SA$, right triangles SEB and SFA are equal, and $\angle ASF (=\angle ASH) = \angle BSE$.

Now, since $\angle$'s CSD, CSE, and CSH are right angles, $\angle CSD = \frac{1}{2}(\angle CSE + \angle SCH)$, $= \frac{1}{2}(\angle CSB + \angle BSE + \angle CSA - \angle ASH)$
$= \frac{1}{2}(\angle CSB + \angle CSA)$.

Case III. $\angle DSC >$a right angle. (See Fig. 3.)

To prove $\angle DSC > \frac{1}{2}(\angle CSA + \angle CSB)$.

Produce AS, BS, and DS, forming another trihedral angle $S-A'B'C$.

By Case I, $\angle CSD' < \frac{1}{2}(CSA'(\angle CSA' + \angle CSB')$, or $180^\circ - \angle CSD < \frac{1}{2}(180^\circ - \angle CSA + 180^\circ - \angle CSB)$, from which $-\angle CSD < -\frac{1}{2}(\angle CSA + \angle CSB)$.

Therefore, $\angle CSD > \frac{1}{2}(\angle CSA + \angle CSB)$.

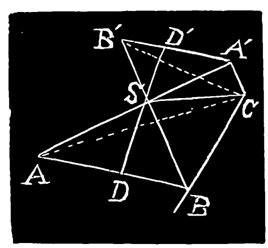


Fig. 3.

Also solved by *H. C. WHITAKER*.

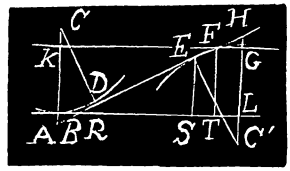
155. Proposed by J. C. NAGLE, M.A., C.E., Professor of Civil Engineering, State Agricultural and Mechanical College, College Station, Texas.

A special case of the following problem was sent me some time ago by an ex-member of one of my engineering classes, as occurring on the Southern Pacific Ry., near Devil's River:

Two straight tracks, p feet between centers, are to be united by a cross-over composed of two curves of radius R, and a length L of intervening tangent. Required the central angles and the distance between tangent points, measured along main track. In the special case referred to p was 62 feet, L 100 feet with 9° 30 curves.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $\tan OCD=\tan EC'G=m$, $OC=CD=C'G=C'E=R$, $DE=L$, $KO=LG=p$. The equation to the line AH is $y=mx-b$, where O is origin and $b=OA$.



$CA=CD\sec OCD$. $\therefore R+b=R\sqrt{(1+m^2)}$.

$\therefore y=mx+R-R\sqrt{(1+m^2)}$.

When $y=0$, $x=R[\sqrt{(1+m^2)}-1]/m$,

When $y=p$, $x=[p+R\sqrt{(1+m^2)}-R]/m=x_1$.

$\therefore OB=BD=EF=FG=R[\sqrt{(1+m^2)}-1]/m$.

Since $BF=\sqrt{[BT^2+TF^2)}=\sqrt{[(x_1-x)^2+p^2]}$,

$\therefore L+2R[\sqrt{(1+m^2)}-1]/m=\sqrt{[(p/m)^2+p^2]}=(p/m)\sqrt{(1+m^2)}$.

$\therefore [(2R-p)^2-L^2]m^2+4RLm=4Rp-p^2$. This determines m.

$OL=OT+TL=x+x_1=[p+2R\sqrt{(1+m)^2}-2R]/m$.

$BR=BD\cos OCD=x/\sqrt{(1+m^2)}=R[\sqrt{(1+m^2)}-1]/[m\sqrt{(1+m^2)}]$.

$RS=BT-2BR=p/m-2R[\sqrt{(1+m^2)}-1]/[m\sqrt{(1+m^2)}]$.

$p=62$, $L=100$, $R=604$.

$\therefore 325829m^2+60400m=36487$ or $m=.254559$.

$\therefore \angle OCD=14° \; 16' \; 52.6''=$central angles.

$OL=394.9$ feet, $RS=96.9$ feet.

CALCULUS.

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112. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A sphere of radius r is pierced by a cylinder radius $\frac{1}{2}r$ so that the cylinder just grazes the center of the sphere. Find volume removed; the lateral surface and the spherical surface removed.

Solution by L. C. WALKER, A. M., Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.; G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; and the PROPOSER.

Taking the center of the sphere for coördinates, we have for the equations of sphere and cylinder, respectively,

$$x^2+y^2+z^2=r^2,\ y^2=rx-x^2.$$

Therefore volume removed, $V=4\int_0^r dx\int_0^{\sqrt{(rx-x^2)}} zdy$

$$=2\int_0^r\left[(r^2-x^2)\sin^{-1}\sqrt{\frac{x}{r+x}}dx+\sqrt{(rx)(r-x)}dx\right]=\frac{3\pi-4}{9}.2r^3.$$

The lateral surface $S=2\int z\partial s$, s being an arc of the base circle of the cylinder. From $y^2=rx-x^2$,